

# Kadell's two conjectures for Macdonald polynomials

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Recently Kevin Kadell found interesting properties of anti-symmetric variants of the so-called Jack polynomials [Ka]. He formulated two conjectures about negative integral and half-integral values of the parameter  $k$  ( $k = 1$  for the characters of compact simple Lie groups). As it was observed independently by Ian Macdonald and the author, these conjectures follow readily from the interpretation of the Jack polynomials as eigenfunctions of the Calogero-Sutherland -Heckman -Opdam second order operators generalizing the radial parts of the Laplace operators on symmetric spaces (see [HO,He,M1,M2]). The difference case requires a bit different treatment but still is not complicated. We will formulate and prove the Kadell conjectures for the Macdonald polynomials (the  $q, t$ -case).

These statements are of certain interest because negative  $k$  are somehow connected with irreducible representations for anti-dominant highest weights (and with representations of Kac-Moody algebras of negative integral central charge). They also make more complete the theory of Macdonald's polynomials at roots of unity started in [Ki], [C3,C4]. Half-integral  $k$  appear in the theory of spherical functions. Anyway it is challenging to understand what is going on when  $0 > k \in \mathbf{Q}$ , since these values are singular for the coefficients of symmetric Macdonald polynomials.

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## 1. Jack polynomials

Let  $R = \{\alpha\} \subset \mathbf{R}^n$  be a root system of type  $A, B, \dots, F, G$  with respect to a euclidean form  $(z, z')$  on  $\mathbf{R}^n \ni z, z'$ ,  $W$  the Weyl group generated by the reflections  $s_\alpha$ . We assume that  $(\alpha, \alpha) = 2$  for long  $\alpha$ . Let us fix the set  $R_+$  of positive roots ( $R_- = -R_+$ ), the corresponding simple roots  $\alpha_1, \dots, \alpha_n$ , and their dual counterparts  $a_1, \dots, a_n, a_i = \alpha_i^\vee$ , where  $\alpha^\vee = 2\alpha/(\alpha, \alpha)$ . The dual

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fundamental weights  $b_1, \dots, b_n$  are determined from the relations  $(b_i, \alpha_j) = \delta_i^j$  for the Kronecker delta. We will also introduce the dual root system  $R^\vee = \{\alpha^\vee, \alpha \in R\}$ ,  $R_+^\vee$ , and the lattices

$$A = \oplus_{i=1}^n \mathbf{Z} a_i \subset B = \oplus_{i=1}^n \mathbf{Z} b_i,$$

$A_+, B_+$  for  $\mathbf{Z}_+ = \{m \in \mathbf{Z}, m \geq 0\}$  instead of  $\mathbf{Z}$ . (In the standard notations,  $A = Q^\vee$ ,  $B = P^\vee$  - see [B].) Later on,

$$(1.1) \quad \begin{aligned} \nu_\alpha &= \nu_{\alpha^\vee} = (\alpha, \alpha), \quad \nu_i = \nu_{\alpha_i}, \quad \nu_R = \{\nu_\alpha, \alpha \in R\}, \\ \rho_\nu &= (1/2) \sum_{\nu_\alpha = \nu} \alpha = (\nu/2) \sum_{\nu_i = \nu} b_i, \quad \text{for } \alpha \in R_+, \\ r_\nu &= \rho_\nu^\vee = (2/\nu) \rho_\nu = \sum_{\nu_i = \nu} b_i, \quad 2/\nu = 1, 2, 3. \end{aligned}$$

We set  $x_i = \exp(b_i)$ ,  $x_b = \exp(b) = \prod_{i=1}^n x_i^{\kappa_i}$  for  $b = \sum_{i=1}^n \kappa_i b_i$ ,  $\mathbf{C}[x] \stackrel{\text{def}}{=} \mathbf{C}[x_b, b \in B]$ ,  $\partial_a(x_b) = (a, b)x_b$ . The monomial symmetric functions  $m_b = \sum_{c \in W(b)} x_{-c}$  for  $b \in B_+$  form a base of the space  $\mathbf{C}[x]^W$  of all  $W$ -invariant polynomials (note the sign of  $c$ ).

Let  $k_\nu \in \mathbf{C}$ ,  $\nu \in \nu_R$ ,  $k_\alpha = k_{\nu_\alpha} = k_{\alpha^\vee}$ ,  $r_k = \sum_\nu k_\nu r_\nu$ . The operator

$$(1.2) \quad L_2^{(k)} = \sum_{i=1}^n \partial_{\alpha_i} \partial_{b_i} + \sum_{a \in R_+^\vee} k_a \frac{x_a + 1}{x_a - 1} \partial_a + (r_k, r_k),$$

can be also represented as  $L_2^{(k)} = \Delta_k^{-1} H_2^{(k)} \Delta_k$  for

$$(1.3) \quad \begin{aligned} H_2^{(k)} &= \sum_{i=1}^n \partial_{\alpha_i} \partial_{b_i} + \sum_{a \in R_+^\vee} k_a (1 - k_a) (a, a) (\exp(a/2) - \exp(-a/2))^{-2}, \\ \Delta_k &= \prod_{a \in R_+^\vee} (\exp(a/2) - \exp(-a/2))^{k_a}. \end{aligned}$$

Here  $\Delta_k$  for non-integral  $k$  should be a solution of the obvious system of differential equations. The passage from  $L$  to  $H$  is well known (see e.g. [C1], Corollary 2.8).

Jack polynomials  $j_b^{(k)}(x) \in \mathbf{C}[x]$ ,  $b \in B_+$  belong to the algebra  $\mathbf{C}[x]^W$  of  $W$ -invariant polynomials. They can be fixed up to proportionality from the following eigenvalue problem:

$$(1.4) \quad L_2^{(k)}(j_b^{(k)}) = (b + r_k, b + r_k) j_b^{(k)}.$$

More exactly (see e.g. [He, M2, O]), they are determined uniquely by means of this equation and the conditions

$$(1.5) \quad \begin{aligned} J_b^{(k)} - m_b &\in \oplus_c \mathbf{C}(q, t) m_c \quad \text{for } c \prec b \\ \text{where } c \in B_+, \quad c \prec b &\text{ means that } b - c \in A_+, c \neq b. \end{aligned}$$

Here  $k_\nu$  are arbitrary except negative rational numbers.

A more traditional approach is as follows (see also [M1] and the papers by Hanlon, Stanley). Let  $\langle f \rangle$  be the constant term of  $f \in \mathbf{C}[x]$ . Setting for  $k_\nu \in \mathbf{Z}$ ,

$$(1.6) \quad \langle f, g \rangle = \langle \Delta_{2k} f(x) g(x^{-1}) \rangle = \langle g, f \rangle \quad \text{where } f, g \in \mathbf{C}(q, t)[x]^W,$$

one can introduce  $J_b^{(k)}$  by means of the conditions  $\langle j_b^{(k)}, m_c \rangle = 0$ , for  $c \prec b$  together with (1.5). They are pairwise orthogonal for arbitrary  $b \in B_+$ , since  $H_2^{(k)}$  is self-adjoint and therefore  $L_2^{(k)}$  is self-adjoint with respect to the pairing  $\langle \cdot, \cdot \rangle$  (the eigenvalues distinguish  $b$  at least for non-negative  $k$ ). If  $k$  is not an integer then the pairing should be understood analytically.

THEOREM (KADELL CONJECTURES) 1.1. *Let  $m_\nu \in \mathbf{Z}_+$ . Then*

$$\Delta_{2m} j_b^{(1/2+m)} = j_{b+2r_m}^{(1/2-m)}, \quad b \in B_+.$$

*The polynomial  $\Delta_{2m+1} j_b^{(m+1)}$  is anti-symmetric ( $s_i(\cdot) = -(\cdot)$ ,  $1 \leq i \leq n$ ) and satisfies (1.4) for  $k = -m$  and the eigenvalue equal to  $(b + r_{m+1}, b + r_{m+1})$ .*

The proof results immediately from the invariance of  $H_2^{(k)}$  when  $k$  is replaced by  $1 - k$ .

## 2. Macdonald polynomials

Let us introduce the algebra  $\mathbf{C}(q, t)[x]$  of polynomials in terms of  $x_i^{\pm 1}$  with the coefficients belonging to the field  $\mathbf{C}(q, t)$  of rational functions in terms of indefinite complex parameters  $q, t_\nu, \nu \in \nu_R$  (we will put  $t_\alpha = t_{\nu_\alpha} = t_{\alpha^\vee}$ ). The coefficient of  $x^0 = 1$  (the constant term) will be again denoted by  $\langle \cdot \rangle$ . The following product is a Laurent series in  $x$  with the coefficients in the algebra  $\mathbf{C}[t][[q]]$  of formal series in  $q$  over polynomials in  $t$ :

$$(2.1) \quad \mu = \mu_{q,t} = \prod_{a \in R_+^\vee} \prod_{i=0}^{\infty} \frac{(1 - x_a q_a^i)(1 - x_a^{-1} q_a^{i+1})}{(1 - x_a t_a q_a^i)(1 - x_a^{-1} t_a q_a^{i+1})},$$

where  $q_a = q_\nu = q^{2/\nu}$  for  $\nu = \nu_a$ . We note that  $\mu \in \mathbf{C}(q, t)[x]$  if  $t_\nu = q_\nu^{k_\nu}$  for  $k_\nu \in \mathbf{Z}_+$ . The coefficients of  $\mu_1 \stackrel{\text{def}}{=} \mu / \langle \mu \rangle$  are from  $\mathbf{C}(q, t)$ , where the formula for the constant term of  $\mu$  is as follows (see [C2]):

$$(2.2) \quad \langle \mu \rangle = \prod_{a \in R_+^\vee} \prod_{i=1}^{\infty} \frac{(1 - x_a(t^\rho) q_a^i)^2}{(1 - x_a(t^\rho) t_a q_a^i)(1 - x_a(t^\rho) t_a^{-1} q_a^i)}.$$

We note that  $\mu_1^* = \mu_1$  with respect to the involution

$$x_b^* = x_{-b}, \quad t^* = t^{-1}, \quad q^* = q^{-1}.$$

Setting ,

$$(2.3) \quad \langle f, g \rangle = \langle \mu_1 f, g^* \rangle = \langle g, f \rangle^* \text{ for } f, g \in \mathbf{C}(q, t)[x]^W,$$

we introduce the Macdonald polynomials  $p_b^{q,t} = p_b(x)$ ,  $b \in B_+$ , by means of the conditions

$$(2.4) \quad p_b - m_b \in \oplus_c \mathbf{C}(q, t)m_c, \quad \langle p_b, m_c \rangle = 0, \text{ for } c \prec b.$$

They can be determined by the Gram - Schmidt process because the (skew Macdonald) pairing (see [M1,M2,C2]) is non-degenerate (for generic  $q, t$ ) and form a basis in  $\mathbf{C}(q, t)[x]^W$ . As it was established by Macdonald they are pairwise orthogonal for arbitrary  $b \in B_+$ . We note that  $p_b$  are "real" with respect to the formal conjugation sending  $q \rightarrow q^{-1}$ ,  $t \rightarrow t^{-1}$ . It makes our definition compatible with Macdonald's original one (his  $\mu$  is somewhat different).

The construction is applicable when  $t_\nu = q_\nu^{k_\nu}$  for  $k_\nu \in \mathbf{Z}_+$ . More exactly, the formulas for the coefficients of Macdonald polynomials have singularities only for rational negative  $k$  (if  $q$  is generic). We come to the operator interpretation of the Macdonald polynomials.

### 3. Affine root systems

The vectors  $\tilde{\alpha} = [\alpha, k] \in \mathbf{R}^n \times \mathbf{R} \subset \mathbf{R}^{n+1}$  for  $\alpha \in R, k \in \mathbf{Z}$  form the affine root system  $R^a \supset R$  ( $z \in \mathbf{R}^n$  are identified with  $[z, 0]$ ). We add  $\alpha_0 \stackrel{def}{=} [-\theta, 1]$  to the simple roots for the maximal root  $\theta \in R$ . The corresponding set  $R_+^a$  of positive roots coincides with  $R_+ \cup \{[\alpha, k], \alpha \in R, k > 0\}$ . See [B].

We denote the Dynkin diagram and its affine completion with  $\{\alpha_j, 0 \leq j \leq n\}$  as the vertices by  $\Gamma$  and  $\Gamma^a$ . Let  $m_{ij} = 2, 3, 4, 6$  if  $\alpha_i$  and  $\alpha_j$  are joined by 0, 1, 2, 3 laces respectively. The set of the indices of the images of  $\alpha_0$  by all the automorphisms of  $\Gamma^a$  will be denoted by  $O$  ( $O = \{0\}$  for  $E_8, F_4, G_2$ ). Let  $O^* = r \in O, r \neq 0$ .

Given  $\tilde{\alpha} = [\alpha, k] \in R^a$ ,  $b \in B$ , let

$$(3.1) \quad s_{\tilde{\alpha}}(\tilde{z}) = \tilde{z} - (z, \alpha^\vee)\tilde{\alpha}, \quad b'(\tilde{z}) = [z, \zeta - (z, b)]$$

for  $\tilde{z} = [z, \zeta] \in \mathbf{R}^{n+1}$ .

The affine Weyl group  $W^a$  is generated by all  $s_{\tilde{\alpha}}$ . One can take the simple reflections  $s_j = s_{\alpha_j}, 0 \leq j \leq n$ , as its generators and introduce the corresponding notion of the length. This group is the semi-direct product  $W \ltimes A'$  of its subgroups  $W$  and  $A' = \{a', a \in A\}$ , where

$$(3.2) \quad a' = s_\alpha s_{[\alpha, 1]} = s_{[-\alpha, 1]} s_\alpha \text{ for } a = \alpha^\vee, \alpha \in R.$$

The extended Weyl group  $W^b$  generated by  $W$  and  $B'$  (instead of  $A'$ ) is isomorphic to  $W \ltimes B'$ :

$$(3.3) \quad (wb')([z, \zeta]) = [w(z), \zeta - (z, b)] \text{ for } w \in W, b \in B.$$

Given  $b \in B_+$ , let

$$(3.4) \quad \omega_b = w_0 w_0^+ \in W, \quad \pi_b = b'(\omega_{b_+})^{-1} \in W^b, \quad \pi_i = \pi_{b_i},$$

where  $w_0$  (respectively,  $w_0^+$ ) is the longest element in  $W$  (respectively, in  $W_b$  generated by  $s_i$  preserving  $b$ ) relative to the set of generators  $\{s_i\}$  for  $i > 0$ .

We will use here only the elements  $\pi_r = \pi_{b_r}, r \in O$ . They leave  $\Gamma^a$  invariant and form a group denoted by  $\Pi$ , which is isomorphic to  $B/A$  by the natural projection  $\{b_r \rightarrow \pi_r\}$ . The relations  $\pi_r(\alpha_0) = \alpha_r$  separate the indices  $r \in O^*$  (see e.g. [C2]), and

$$(3.5) \quad W^b = \Pi \ltimes W^a, \quad \text{where } \pi_r s_i \pi_r^{-1} = s_j \text{ if } \pi_r(\alpha_i) = \alpha_j, \quad 0 \leq j \leq n.$$

Given  $\nu \in \nu_R$ ,  $r \in O^*$ ,  $\tilde{w} \in W^a$ , and a reduced decomposition  $\tilde{w} = s_{j_l} \dots s_{j_2} s_{j_1}$  with respect to  $\{s_j, 0 \leq j \leq n\}$ , we call  $l = l(\tilde{w})$  the length of  $\hat{w} = \pi_r \tilde{w} \in W^b$ .

Later on  $b$  and  $b'$  will not be distinguished. We set  $([a, k], [b, l]) = (a, b)$  for  $a, b \in B$ ,  $a_0 = \alpha_0$ ,  $\nu_{\alpha^\vee} = \nu_\alpha$ .

#### 4. Difference operators

We put  $m = 2$  for  $D_{2k}$  and  $C_{2k+1}$ ,  $m = 1$  for  $C_{2k}, B_k$ , otherwise  $m = |\Pi|$ . Later on  $\mathbf{C}_q$  is the field of rational functions in  $q^{1/m}$ . Setting

$$(4.1) \quad x_{\tilde{b}} = \prod_{i=1}^n x_i^{k_i} q^k \text{ if } \tilde{b} = [b, k], b = \sum_{i=1}^n k_i b_i \in B, \quad k \in \frac{1}{m} \mathbf{Z},$$

we will identify polynomials with the corresponding multiplication operators in  $\mathbf{C}_q[x] = \mathbf{C}_q[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ . We replace  $\mathbf{C}_q$  by  $\mathbf{C}_{q,t}$  if the functions (coefficients) also depend rationally on  $\{t_\nu^{1/2}\}$ .

The elements  $\hat{w} \in W^b$  act in  $\mathbf{C}_q[x]$  by the formulas:

$$(4.2) \quad \hat{w}(x_{\tilde{b}}) = x_{\hat{w}(\tilde{b})}.$$

In particular:

$$(4.3) \quad \pi_r(x_b) = x_{\omega_r^{-1}(b)} q^{(b_{r^*}, b)} \text{ for } \alpha_{r^*} = \pi_r^{-1}(\alpha_0), \quad r \in O^*.$$

The Demazure-Lusztig operators (see [C1, C2] for more detail )

$$(4.4) \quad T_j^{q,t} = t_j^{1/2} s_j + (t_j^{1/2} - t_j^{-1/2})(x_{a_j} - 1)^{-1}(s_j - 1), \quad 0 \leq j \leq n.$$

act in  $\mathbf{C}_{q,t}[x]$  naturally. We note that only  $\hat{T}_0$  depends on  $q$ :

$$(4.5) \quad T_0 = t_0^{1/2} s_0 + (t_0^{1/2} - t_0^{-1/2})(q x_\theta^{-1} - 1)^{-1}(s_0 - 1),$$

where  $s_0(x_i) = x_i x_\theta^{-(b_i, \theta)} q^{(b_i, \theta)}.$

Given  $\tilde{w} \in W^a, r \in O$ , the product

$$(4.6) \quad T_{\pi_r \tilde{w}} \stackrel{\text{def}}{=} \pi_r \prod_{k=1}^l T_{i_k}, \quad \text{where } \tilde{w} = \prod_{k=1}^l s_{i_k}, l = l(\tilde{w}),$$

does not depend on the choice of the reduced decomposition (because  $\{T\}$  satisfy the same “braid” relations as  $\{s\}$  do). Moreover,

$$(4.7) \quad T_{\hat{v}} T_{\hat{w}} = T_{\hat{v}\hat{w}} \quad \text{whenever } l(\hat{v}\hat{w}) = l(\hat{v}) + l(\hat{w}) \quad \text{for } \hat{v}, \hat{w} \in W^b.$$

In particular, we arrive at the pairwise commutative elements

$$(4.8) \quad Y_b = \prod_{i=1}^n Y_i^{k_i} \quad \text{if } b = \sum_{i=1}^n k_i b_i \in B, \quad \text{where } Y_i \stackrel{\text{def}}{=} T_{b_i},$$

satisfying the relations

$$(4.9) \quad \begin{aligned} T_i^{-1} Y_b T_i^{-1} &= Y_b Y_{a_i}^{-1} \quad \text{if } (b, \alpha_i) = 1, \\ T_i Y_b &= Y_b T_i \quad \text{if } (b, \alpha_i) = 0, \quad 1 \leq i \leq n. \end{aligned}$$

Given an operator

$$(4.10) \quad H = \sum_{b \in B, w \in W} h_{b,w} b w,$$

where  $h_{b,w}$  belong to the field  $\mathbf{C}_{q,t}(x)$  of rational functions in  $x_1, \dots, x_n$ , we set

$$(4.11) \quad [H]_{\dagger} = \sum h_{b,w} b.$$

We will use the following theorem from [C2,C3] (see also [M3]).

**THEOREM 4.1.** (i) *The difference operators  $\{L_f^{q,t} \stackrel{\text{def}}{=} [f(Y_1, \dots, Y_n)]_{\dagger} \text{ for } f \in \mathbf{C}[x]^W\}$  are pairwise commutative,  $W$ -invariant (i.e.  $w L_f w^{-1} = L_f$  for all  $w \in W$ ) and preserve  $\mathbf{C}_{q,t}[x]^W$ . The Macdonald polynomials  $p_b = p_b^{q,t} (b \in B_+)$  from (2.4) are their eigenvectors:*

$$(4.12) \quad L_f(p_b^{q,t}) = f(t^\rho q^b) p_b^{q,t}, \quad x_i(t^\rho q^b) \stackrel{\text{def}}{=} q^{(b_i, b)} \prod_{\nu} t_{\nu}^{(b_i, \rho \nu)}.$$

(ii) *Given  $v \in \nu_R$  such that  $\nu \in v$  if  $t_{\nu} \neq 1$ , we set*

$$d_v^{q,t} = \prod_{\nu_a \in v} ((t_a x_a)^{1/2} - (t_a x_a)^{-1/2}), \quad t q_v = \{t_{\nu} \text{ if } \nu \notin v, \quad t_{\nu} q^{2/\nu} \text{ otherwise}\}.$$

*The polynomials  $g_b = g_b^{q,t,v} = d_v^{q,t} p_b^{q,tq_v}$  satisfy the relations*

$$(4.13) \quad \begin{aligned} f(Y^{q,t})(g_b) &= f((t q_v)^{\rho} q^b) g_b \quad \text{for } f \in \mathbf{C}[x]^W, \\ T_i^t(g_b) &= \varepsilon t_i^{\varepsilon 1/2} g_b, \quad 1 \leq i \leq n, \end{aligned}$$

*where  $\varepsilon = \pm$  if  $\nu_i \notin v$ ,  $\nu_i \in v$  respectively.*

### 5. Kadell's formulas

The relations

$$(5.1) \quad (s_i)^\iota = -s_i, \quad x_i^\iota = x_i, \quad 0 \leq i \leq n, \quad q^\iota = q, \quad (t^{1/2})^\iota = -t^{-1/2}$$

can be extended to an involution on functions of  $x$  and operators. We will also use the conjugation which sends  $t^{1/2}$  to  $-t^{1/2}$  (and does not change other generators of  $\mathbf{C}_{q,t}[x]$ ). The latter fixes the Macdonald polynomials since they are expressed via  $t$  only. We write  $p^{(k)}, L^{(k)}$  and so on if  $t_\nu = q_\nu^{k_\nu}$ .

MAIN THEOREM 5.1. (i) Let  $m_\nu \in \mathbf{Z}_+$ ,  $t_\nu = q_\nu^{m_\nu+1/2}$  ( $\nu \in \nu_R$ ). Then

$$(5.2) \quad \begin{aligned} \delta_{2m} p_b^{(1/2+m)} &= p_{b+2r_m}^{(1/2-m)} \quad \text{for} \\ \delta_{2m} &= \prod_{a \in R_+^\vee} \{ (x_a^{1/2} t_a^{1/2} q_a^{-1/2} - x_a^{-1/2} t_a^{-1/2} q_a^{1/2}) \cdots \\ &\quad (x_a^{1/2} t_a^{1/2} q_a^{-i/2} - x_a^{-1/2} t_a^{-1/2} q_a^{i/2}) \cdots (x_a^{1/2} t_a^{-1/2} q_a^{1/2} - x_a^{-1/2} t_a^{1/2} q_a^{-1/2}) \}. \end{aligned}$$

(ii) If  $\delta_{2m+1}$  is given by the same formula for  $t_\nu = q_\nu^{m_\nu+1}$ , then the polynomials  $\delta_{2m+1} p_b^{(m+1)}$  are anti-symmetric ( $s_i(\cdot) = -(\cdot)$  where  $1 \leq i \leq n$ ) and satisfy relations (4.12) for  $t'_\nu = q_\nu^{-m_\nu}$  and the eigenvalues equal to  $f(t^\rho q^b)$ .

The proof will be based on the following formulas (for arbitrary  $q, t$ ) which are checked exactly in the same way as the properties of the main anti-involution from [C2] (Theorem 4.1):

$$(5.3) \quad (T_j^{q,t})^\iota = \mu_{q,t} T_j^{q,t} \mu_{q,t}^{-1}, \quad 0 \leq j \leq n.$$

We see that the multiplication by  $\mu_{q,t}$  takes the eigenfunctions of the operators  $Y_1^{q,t}, \dots, Y_n^{q,t}$  to the eigenfunctions of  $\{(Y_i^{q,t})^\iota\}$  corresponding to the same eigenvalues. For instance,  $g_b^{q,t}$  from Theorem 4.1 calculated for  $v = \nu_R$  will go to an anti-symmetric function  $g'_b$ . Indeed, the latter satisfies relations (4.13) for  $\varepsilon = -t^{-1}$  and  $T^\iota$  instead of  $T$ . It means that it is anti-symmetric. Combining this with statement (ii) of the same theorem we see that the multiplication by  $\mu_{q,t} d^{q,t}$  transfers the polynomials  $p_b^{q,qt}$  to anti-symmetric eigenfunctions of the operators  $L_f^{q,t^{-1}}$ . Here we used that the restriction of  $(-s_i)$  to anti-symmetric functions coincides with  $[\cdot]_\dagger$  which is the restriction of  $s_i$  to symmetric ones ( $i > 0$ ). We can replace  $t^{1/2}$  by  $-t^{1/2}$  because  $\{p_b\}$  depend only on  $t$ .

Substituting  $tq^{-1}$  for  $t$  we obtain exactly assertion (ii). Here  $\mu_{q,t}$  is a polynomial.

Coming to (i), we will use that for  $t = q^{1/2+m}$  the function

$$(5.4) \quad \mu_{q,tq^{-1}}^{-1} \prod_{a \in R_+^\vee} \prod_{i=2}^{2m} \{ (x_a^{1/2} t_a^{1/2} q_a^{-i/2} - x_a^{-1/2} t_a^{-1/2} q_a^{i/2}) \}$$

is anti-symmetric for all  $s_0, s_1, \dots, s_n$ . Hence the corresponding multiplication turns  $s_i$  into  $-s_i$ . Composing this with the multiplication by  $\mu_{q,tq^{-1}} d^{q,tq^{-1}}$  we get (i).

We mention that the statements can be easily extended to the case of an arbitrary set  $v$  (see Theorem 4.1). Replacing  $a \in R^\vee, B, r, q_a$  by  $\alpha \in R, P, \rho, q$  one arrives at the dual counterpart of this construction. We also note that the above reasoning and Proposition 4.2 from [C2] give that for arbitrary  $q, t$  and  $f \in \mathbf{C}[x]^W$

$$(5.5) \quad \psi_{q,t} L_f^{q,t} \psi_{q,t}^{-1} = L_{\bar{f}}^{q,tq^{-1}}, \quad \bar{f}(x) = f(x^{-1}),$$

$$\psi_{q,t} = \prod_{a \in R_+^\vee} (x_a^{1/2} - x_a^{-1/2})^{-1} \prod_{i=0}^{\infty} \frac{(1 - x_a q_a^i)(1 - x_a^{-1} q_a^i)}{(1 - x_a t_a q_a^i)(1 - x_a^{-1} t_a q_a^i)}.$$

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